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A refined factorization of the exponential law

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Let ξ be a (possibly killed) subordinator with Laplace exponent ϕ and denote by $I_\phi = \int_0^\infty e^{-\xi_s} ds$, the so-called exponential functional. Consider the positive random variable I_{ψ_1} whose law, according to Bertoin and Yor [*Electron. Comm. Probab.* **6** (2001) 95–106], is determined by its negative entire moments as follows:

$$\mathbb{E}[I_{\psi_1}^{-n}] = \prod_{k=1}^n \phi(k), \quad n = 1, 2, \dots$$

In this note, we show that I_{ψ_1} is a positive self-decomposable random variable whenever the Lévy measure of ξ is absolutely continuous with a monotone decreasing density. In fact, I_{ψ_1} is identified as the exponential functional of a spectrally negative (sn , for short) Lévy process. We deduce from Bertoin and Yor [*Electron. Comm. Probab.* **6** (2001) 95–106] the following factorization of the exponential law \mathbf{e} :

$$I_\phi / I_{\psi_1} \stackrel{(d)}{=} \mathbf{e},$$

where I_{ψ_1} is taken to be independent of I_ϕ . We proceed by showing an identity in distribution between the entrance law of an sn self-similar positive Feller process and the reciprocal of the exponential functional of sn Lévy processes. As a by-product, we obtain some new examples of the law of the exponential functionals, a new factorization of the exponential law and some interesting distributional properties of some random variables. For instance, we obtain that $S(\alpha)^\alpha$ is a self-decomposable random variable, where $S(\alpha)$ is a positive stable random variable of index $\alpha \in (0, 1)$.

Keywords: exponential functional; Lévy processes; self-decomposable random variable; self-similar Markov process; Stieltjes moment sequences; subordinator

1. Introduction

Let $\xi = (\xi_t, t \geq 0)$ be a possibly killed subordinator starting from 0, that is, a $[0, \infty)$ -valued (∞ serves as absorbing state) Lévy process such that $\xi_0 = 0$. The law of ξ is

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well known to be characterized by its Laplace exponent ϕ , which admits the following Lévy–Khintchine representation for any $u \geq 0$:

$$\phi(u) = bu + \int_0^\infty (1 - e^{-ur})\nu(dr) + q, \quad (1.1)$$

where $q \geq 0$ is the killing rate, $b \geq 0$ is the drift and the Lévy measure ν satisfies the integrability condition $\int_{\mathbb{R}^+} (1 \wedge r)\nu(dr) < \infty$. Note that functions of the form (1.1) are also named, in the literature, as Bernstein functions. We refer to the monographs [5, 16] (resp., [4, 14]) for a detailed account on Lévy processes (resp., Bernstein functions). Next, consider the so-called exponential functional associated to ξ , which is defined as

$$I_\phi = \int_0^{\mathbf{e}_q} e^{-\xi_s} ds,$$

where \mathbf{e}_q is an independent exponential random variable with parameter q and we understand that $\mathbf{e}_0 = +\infty$. Note that, for any $q \geq 0$, $I_\phi < \infty$ a.s. We refer to the survey paper of Bertoin and Yor [9] for a thorough description of the properties of this positive random variable and of the motivations for studying its law. In particular, we mention that the law of I_ϕ has been determined through its positive entire moments by Carmona *et al.* [11] as follows:

$$\mathbb{E}[I_\phi^n] = \frac{\Gamma(n+1)}{\prod_{k=1}^n \phi(k)}, \quad n = 1, 2, \dots, \quad (1.2)$$

where Γ stands for the gamma function. Bertoin and Yor [6] (see also [9], Theorem 2, for the case $q > 0$) showed that there exists a positive random variable J whose law is determined by its positive entire moments as follows:

$$\mathbb{E}[J^n] = \prod_{k=1}^n \phi(k), \quad n = 1, 2, \dots,$$

such that, when J is taken to be independent of I_ϕ , one has the following factorization of the exponential law:

$$I_\phi J \stackrel{(d)}{=} \mathbf{e}, \quad (1.3)$$

where “ $\stackrel{(d)}{=}$ ” means identity in distribution and $\mathbf{e} = \mathbf{e}_1$.

Let us now point out that since the random variable J is defined on the half-line and its law is uniquely determined by its positive entire moments, the sequence $(s_n = \prod_{k=1}^n \phi(k))_{n \geq 0}$ corresponds to a determinate normalized Stieltjes moment sequence. In this direction, we should mention that Berg [1] generalizes the above fact by showing that for any $c > 0$, the sequence $(s_n^c)_{n \geq 0}$ associated to a measure on the half-line ρ_c is also a Stieltjes moment sequence which is determinate for $c \leq 2$. He then deduces that there exists a unique product convolution semigroup $(\rho_c)_{c > 0}$ such that the moments of ρ_c

are given by s_n^c for any $c > 0$. Moreover, in [2], Berg characterizes the set of normalized Stieltjes moment sequences for which this power stability property still holds. In the same vein, Berg and Durán [3] study a more general mapping which allows, in particular, the construction of a Stieltjes moment sequence of the form $(s_n)_{n \geq 0}$ with the Bernstein function ϕ replaced by a completely monotone function.

The first aim of this note is to show that the random variable $1/J$ is actually a positive self-decomposable random variable, provided that the Lévy measure ν in (1.1) admits a monotone decreasing density. This will be achieved by identifying the random variable $1/J$ as the exponential functional of a spectrally negative Lévy process which we now introduce. Let $\Xi = (\Xi_t, t \geq 0)$ be a conservative spectrally negative Lévy process with a non-negative mean m and starting from 0, that is, a Lévy process having only negative jumps such that $0 \leq m = \mathbb{E}[\Xi_1] < \infty$. Its law is characterized by its Laplace exponent ψ which admits, in this case, the following Lévy–Khintchine representation for any $u \geq 0$:

$$\psi(u) = \sigma u^2 + mu + \int_{-\infty}^0 (e^{ur} - 1 - ur)\Pi(dr), \quad (1.4)$$

where $\sigma \geq 0$ is the Gaussian coefficient and the Lévy measure satisfies the condition $\int_{-\infty}^0 (|r| \wedge r^2)\Pi(dr) < \infty$. The exponential functional associated to Ξ , denoted by I_ψ , is finite a.s. whenever $m > 0$. Its law has been determined through its negative entire moments by Bertoin and Yor [7], as follows:

$$\mathbb{E}[I_\psi^{-n}] = m \frac{\prod_{k=1}^{n-1} \psi(k)}{\Gamma(n)}, \quad n = 1, 2, \dots, \quad (1.5)$$

with the convention that the right-hand side is m when $n = 1$.

We now recall that Lamperti [18], interested in limit theorems for stochastic processes, shows, in particular, that for any $x > 0$, the process $X = (X_t, t \geq 0)$, defined for any $t \geq 0$ by

$$X_t = x \exp(\Xi_{A_t/x}), \quad A_t = \inf \left\{ s \geq 0; \int_0^s e^{\Xi_u} du > t \right\}, \quad (1.6)$$

starting from x at time 0, is a self-similar Feller process on $(0, \infty)$ having only negative jumps. The Lamperti transformation is actually one-to-one and extends to any Lévy process. Bertoin and Yor [7], Proposition 1, shows that the family of probability measures $(\mathbb{Q}_x^{(\psi)})_{x>0}$ of X , as defined in (1.6), converges as $x \downarrow 0$, in the sense of finite-dimensional distributions, to a probability measure $\mathbb{Q}_0^{(\psi)}$; see also [10] for the weak convergence in the Skorokhod topology. Thus, X is a Feller process on $[0, \infty)$ and Bertoin and Yor determine the law of the random variable $J_\psi = (X_1, \mathbb{Q}_0^{(\psi)})$, the entrance law of X at time 1, in terms of its positive entire moments as follows:

$$\mathbb{E}[J_\psi^n] = \frac{\prod_{k=1}^n \psi(k)}{\Gamma(n+1)}, \quad n = 1, 2, \dots \quad (1.7)$$

They also deduce, in the case where $m > 0$ and ξ is the ascending ladder height process of the dual process of Ξ (see, e.g., [5], Chapter VI), that the random variable J , in (1.3), is J_ψ , that is,

$$I_\phi J_\psi \stackrel{(d)}{=} \mathbf{e}. \quad (1.8)$$

The second aim of this note is to relate, in a simple way, the law of J_ψ , for any $m \geq 0$, with the exponential functional of a spectrally negative Lévy process.

Finally, as observed by Rivero [22], the study of the exponential functional is also motivated by its connection to some interesting random equations. Indeed, from the strong Markov property for Lévy processes, which entails that for any finite stopping time T in the natural filtration $(\mathcal{F}_t, t \geq 0)$ of ξ , the process $(\xi_{t+T} - \xi_T, t \geq 0)$ is independent of \mathcal{F}_T and has the same distribution as ξ , we readily deduce that the random variable I_ϕ , in the case $q = 0$, is a solution to the random affine equation

$$I_\phi \stackrel{(d)}{=} \int_0^T e^{-\xi_s} ds + e^{-\xi_T} I'_\phi, \quad (1.9)$$

where, on the right-hand side, I'_ϕ is an independent copy of I_ϕ . Note that this type of random equation have been studied by Kesten [15] and Goldie [13]. By means of a similar argument, but using the absence of positive jumps of Ξ (see [21], Proposition 4, for more details), we get that I_ψ is a solution to the random affine equation, for any $y > 0$,

$$I_\psi \stackrel{(d)}{=} \int_0^{T_y} e^{-\Xi_s} ds + e^{-y} I'_\psi, \quad (1.10)$$

where $T_y = \inf\{s > 0; \Xi_s \geq y\}$ and, on the right-hand side, I'_ψ is an independent copy of I_ψ . Hence, I_ψ is a positive self-decomposable random variable and, in particular, its law is absolutely continuous and unimodal; see, e.g., [23] and [24] for an excellent account of this set of probability measures.

2. Main results

2.1. Factorization of the exponential law with exponential functionals

In this subsection, we suppose that ξ is a subordinator starting from 0 with Laplace exponent given by (1.1). We introduce the following hypothesis on the Lévy measure of ξ .

Assumption 2.1. *There exists a monotone decreasing function f such that $\nu(dx) = f(x) dx$.*

We recall that, under this condition, $-df(x)$ is a Stieltjes measure on $(0, \infty)$. We also use the notation $-df(-x)$ for the image of the positive measure $-df(x)$ under the map $x \mapsto -x$. For instance, if f is, in addition, differentiable, then $-df(-x) = -f'(-x)dx$. We are now ready to derive our refinement of the factorization of the exponential law.

Theorem 2.2. *Let ξ be a subordinator with Laplace exponent ϕ given by (1.1). If Assumption 2.1 holds, then there exists an independent spectrally negative Lévy process with a positive mean and Laplace exponent ψ_1 , analytic in the domain $C = \{u \in \mathbb{C}; \Re(u) > -1\}$, with $\psi_1(-1) = -\phi(0)$, given by*

$$\psi_1(u) = bu^2 + \phi(1)u + \int_{-\infty}^0 (e^{ur} - 1 - ur)\Pi(dr), \quad u \in C,$$

where $\Pi(dr) = e^r(f(-r)dr - df(-r))$ is a Stieltjes measure on $(-\infty, 0)$. Moreover, the law of the positive self-decomposable random variable I_{ψ_1} is determined by its negative entire moments as follows:

$$\mathbb{E}[I_{\psi_1}^{-n}] = \prod_{k=1}^n \phi(k), \quad n = 1, 2, \dots$$

The exponential law admits the following factorization:

$$I_\phi / I_{\psi_1} \stackrel{(d)}{=} \mathbf{e}, \quad (2.1)$$

where \mathbf{e} stands for an exponential random variable of parameter 1.

Conversely, if ψ is of the form (1.4) with $m > 0$ and is analytic in the domain C with $\psi(-1) \leq 0$, then there exists an independent subordinator with Laplace exponent ϕ_{-1} given by

$$\phi_{-1}(u) = -\psi(-1) + \sigma u + \int_0^\infty (1 - e^{-ur})e^r \Pi(-\infty, -r) dr, \quad u \geq 0,$$

such that

$$I_{\phi_{-1}} / I_\psi \stackrel{(d)}{=} \mathbf{e}. \quad (2.2)$$

Remark 2.3. (1) We have several comments to offer on the identity (2.1) when compared to (1.8). First, our hypotheses are slightly less restrictive. Indeed, it is well known (see, e.g., [5], Chapter VI) that the ascending ladder height process of the dual process of Ξ satisfies Assumption 2.1 and is a killed subordinator; thus, in (1.8), q is necessarily positive. More importantly, we have identified the mixture random variable of I_ϕ in the factorization of the exponential law as the reciprocal of a positive self-decomposable random variable. Finally, our identity allows further explicit examples to be obtained for the law of the exponential functional of Lévy processes. All of these facts will be illustrated in Section 3.

(2) The analyticity property of ψ_1 means that the associated spectrally negative Lévy process Ξ^1 admits exponential moments of order $u \geq -1$, that is, for any $u \geq -1$, we have

$$\mathbb{E}[e^{u\Xi_1^1}] < \infty.$$

We shall show, in Proposition 2.4 below, how to construct a spectrally negative Lévy process with such a property from any spectrally negative Lévy process with a non-negative mean.

Proof of Theorem 2.2. Let us write $\psi_1(u) = u\phi(u+1)$. Then, recalling that ϕ is analytic in the right half-plane, we readily deduce that the mapping $u \mapsto \psi_1(u)$ is analytic in C , with $\psi_1(0) = 0$ and $\psi_1(-1) = -\phi(0)$. Let us now show that under Assumption 2.1, ψ_1 is the Laplace exponent of a spectrally negative Lévy process with a positive mean. On one hand, since $r \mapsto f(r)$ is monotone decreasing on $(0, \infty)$, it follows that $\Pi(dr) = e^r(f(-r)dr - df(-r))$ is clearly a Stieltjes measure on $(-\infty, 0)$. On the other hand, by integration by parts and a change of variable, we have, for any $u \geq 0$,

$$\begin{aligned} \psi_1(u) &= u \left(b(u+1) + \int_0^\infty (1 - e^{-(u+1)r}) f(r) dr + q \right) \\ &= bu^2 + \left(b + q + \int_0^\infty (1 - e^{-r}) f(r) dr \right) u + u \int_0^\infty (1 - e^{-ur}) e^{-r} f(r) dr \\ &= bu^2 + \phi(1)u + \int_0^\infty (e^{-ur} - 1 + ur) e^{-r} (f(r) dr - df(r)) \\ &= bu^2 + \phi(1)u + \int_{-\infty}^0 (e^{ur} - 1 - ur) \Pi(dr). \end{aligned}$$

Checking, by integration by parts, that $\int_{-\infty}^0 (|r| \wedge r^2) \Pi(dr) < \infty$, we get that ψ_1 is the Laplace exponent of a spectrally negative Lévy process with a positive mean since $\psi_1(0^+) = \phi(1) > 0$. Then, by means of the identity (1.5), we obtain, for any $n = 1, 2, \dots$,

$$\mathbb{E}[I_{\psi_1}^{-n}] = \phi(1) \frac{\prod_{k=1}^{n-1} \psi_1(k)}{\Gamma(n)} = \phi(1) \frac{\prod_{k=1}^{n-1} k\phi(k+1)}{\Gamma(n)} = \prod_{k=1}^n \phi(k),$$

where we have used the identity $\Gamma(n) = \prod_{k=1}^{n-1} k$. The self-decomposability of I_{ψ_1} was discussed in the Introduction and the factorization of the exponential law follows readily from the independence of the random variables I_{ψ_1} and I_ϕ and the identity (1.2). The converse follows by means of similar reasoning. We only need to check that $\phi_{-1}(u) = \frac{\psi(u-1)}{u-1}$ is the Laplace exponent of a subordinator. From the conditions imposed on ψ , we can easily deduce that ϕ_{-1} is well defined on \mathbb{R}^+ with $\phi_{-1}(0) = -\psi(-1) \geq 0$. Moreover, by integrations by parts, we get

$$\phi_{-1}(u) = \sigma u - (\sigma - m) + \frac{1}{u-1} \int_{-\infty}^0 (e^{(u-1)r} - 1 - (u-1)r) \Pi(dr)$$

$$\begin{aligned}
&= \sigma u - (\sigma - m) - \int_{-\infty}^0 (e^{(u-1)r} - 1) \Pi(-\infty, r) \, dr \\
&= \sigma u - (\sigma - m) - \int_{-\infty}^0 (e^{ur} - 1) e^{-r} \Pi(-\infty, r) \, dr - \int_{-\infty}^0 (e^{-r} - 1 + r) \Pi(dr) \\
&= -\psi(-1) + \sigma u + \int_0^{\infty} (1 - e^{-ur}) e^r \Pi(-\infty, -r) \, dr.
\end{aligned}$$

The proof of the theorem is thus completed. \square

As a direct consequence of Theorem 2.2, we have the following fact. By self-decomposability, the law of I_{ψ_1} is absolutely continuous and thus the random variable J in (1.3) admits a density with respect to the Lebesgue measure, which, according to Bertoin and Yor [6], is a 1-harmonic function for the self-similar process associated to ξ in the Lamperti mapping (1.6). Thus, writing p_1 for the density of I_{ψ_1} , the mapping $x \mapsto x^{-2} p_1(x^{-1})$ on \mathbb{R}^+ is a 1-harmonic function for the self-similar process associated to ξ .

We complete this part with the following observation. Let us suppose that there exists a subordinator with Laplace exponent $\widehat{\phi}$ such that

$$\phi(u) \widehat{\phi}(u) = u, \quad u \geq 0.$$

Under such a condition, ϕ is called a special Bernstein function and we refer to Kyprianou and Rivero [17], and references therein, for more information on this function. Note that such an identity occurs in fluctuation theory for Lévy processes and that sufficient conditions for ϕ to be a special Bernstein function are that Assumption 2.1 holds, $d = 0$ and the mapping $x \mapsto f(x) (\int_x^{\infty} f(y) \, dy)^{-1}$ is a decreasing function on $(0, \infty)$. As remarked in [6], we have, in this case, with the obvious notation,

$$I_{\widehat{\phi}} I_{\phi} \stackrel{(d)}{=} \mathbf{e}.$$

We deduce readily, from Theorem 2.2, that if ϕ is a special Bernstein function satisfying Assumption 2.1, then

$$I_{\widehat{\phi}} \stackrel{(d)}{=} 1/I_{\psi_1}.$$

Moreover, if, in addition, $\widehat{\phi}$ also satisfies Assumption 2.1, then $1/I_{\phi}$ is a positive self-decomposable random variable. Note that if $\widehat{\phi}(0) = 0$, then $I_{\widehat{\phi}}$ is the solution to the random affine equation (1.9) and thus, in this case, solving this equation reduces to solve the random affine equation with constant coefficient (1.10).

2.2. Exponential functionals and entrance laws J_{ψ}

We now assume that Ξ is a spectrally negative Lévy process with a non-negative mean m . Its Laplace exponent has the form (1.4). We recall that the positive random variable

J_ψ is the entrance law at time 1 of the self-similar Feller process associated to Ξ via the Lamperti mapping (1.6). We also mention that, when $m > 0$, Bertoin and Yor [8] show that the distribution of $1/I_\psi$ is the so-called length-biased distribution of J_ψ , that is, using the identity $\mathbb{E}[1/I_\psi] = m$,

$$\mathbb{E}[g(J_\psi)] = m^{-1} \mathbb{E}[1/I_\psi g(1/I_\psi)]$$

for any measurable function $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$. We refine this connection in the following proposition.

Proposition 2.4. *Let ψ be the Laplace exponent of a spectrally negative Lévy process with mean $m \geq 0$. Then, the mapping defined by*

$$\psi_2(u) = \frac{u}{u+1} \psi(u+1)$$

is analytic in $C = \{u \in \mathbb{C}; \Re(u) > -1\}$ and is the Laplace exponent of a spectrally negative Lévy process with a positive mean $\psi(1)$. Moreover, the identity in distribution

$$J_\psi \stackrel{(d)}{=} 1/I_{\psi_2}$$

holds. Consequently, the law of J_ψ is absolutely continuous for any $m \geq 0$.

Proof. First, since it is well known that ψ is analytic in the right half-plane, it is clear that the mapping $\psi_2(u) = \frac{u}{u+1} \psi(u+1)$ is analytic in C . Next, we recall that ψ has the form

$$\psi(u) = \sigma u^2 + mu + \int_{-\infty}^0 (e^{ur} - 1 - ur) \Pi(dr),$$

where $\int_{-\infty}^0 (|r| \wedge r^2) \Pi(dr) < \infty$ and $\sigma \geq 0$. Thus, by means of integration by parts and writing $f(r) = \Pi(-\infty, r)$ for the tail of the Lévy measure, we get

$$\begin{aligned} & \frac{u}{u+1} \psi(u+1) \\ &= \sigma u^2 + (m + \sigma)u + \frac{u}{u+1} \int_{-\infty}^0 (e^{(u+1)r} - 1 - (u+1)r) \Pi(dr) \\ &= \sigma u^2 + (m + \sigma)u - u \int_{-\infty}^0 (e^{(u+1)r} - 1) f(r) dr \\ &= \sigma u^2 + (m + \sigma)u - u \left(\int_{-\infty}^0 (e^{ur} - 1) e^r f(r) dr + \int_{-\infty}^0 (e^r - 1) f(r) dr \right) \quad (2.3) \\ &= \sigma u^2 + \left(m + \sigma + \int_{-\infty}^0 (e^r - 1 - r) \Pi(dr) \right) u \end{aligned}$$

$$\begin{aligned}
& + \int_{-\infty}^0 (e^{ur} - 1 - ur)e^r (f(r) dr + \Pi(dr)) \\
& = \sigma u^2 + \psi(1)u + \int_{-\infty}^0 (e^{ur} - 1 - ur)e^r (f(r) dr + \Pi(dr)),
\end{aligned}$$

where we recognize the Laplace exponent of a spectrally negative Lévy process. Finally, observing that $\lim_{u \rightarrow 0} \frac{d}{du} \frac{u}{u+1} \psi(u+1) = \psi(1) > 0$ since ψ is increasing on $(0, \infty)$, we have, from (1.5) and any $n = 1, 2, \dots$,

$$\mathbb{E}[I_{\psi_2}^{-n}] = \psi(1) \frac{\prod_{k=1}^{n-1} (k/(k+1)) \psi(k+1)}{\Gamma(n)} = \frac{\prod_{k=1}^n \psi(k)}{\Gamma(n+1)} = \mathbb{E}[J_{\psi}^n],$$

where the last identity follows from (1.7). The absolute continuity property of the law of J_{ψ} follows from that of I_{ψ_2} as a self-decomposable random variable. \square

We mention that the random variable J_{ψ} appears in the study of the so-called Ornstein–Uhlenbeck process associated to X . Indeed, if one considers the stochastic process $U = (U_t, t \geq 0)$ defined, for any $t \geq 0$, by

$$U_t = e^{-t} X_{e^t - 1},$$

then U is a stationary Feller process on $[0, \infty)$ and its unique invariant measure is the law of J_{ψ} ; see, for instance, [19], Theorem 1.2. The above proposition tells us that the invariant measure is absolutely continuous for any $m \geq 0$.

We also indicate that the transformation of the Laplace exponent of a spectrally negative Lévy process used in the proof of Proposition 2.4 is a specific instance of more general mappings of characteristic exponents of Lévy processes introduced and studied by Kyprianou and Patie [12].

3. Some examples

In this section, we will make use of the identities presented in Section 2 to obtain new explicit examples of the law of the exponential functional associated to subordinators or spectrally negative Lévy processes, to obtain a new factorization of the exponential law and to prove the self-decomposability property of some positive random variables.

In [6], the authors study the connection between the law of the exponential functional of some subordinators and the following factorization of the exponential law:

$$\mathbf{e}^{\alpha} S(\alpha)^{-\alpha} \stackrel{(d)}{=} \mathbf{e}, \quad (3.1)$$

where $\alpha \in (0, 1)$ and $S(\alpha)$ is a positive α -stable random variable, independent of \mathbf{e} . We split this example into two parts.

On the one hand, the authors show that

$$I_\phi \stackrel{(d)}{=} S(\alpha)^{-\alpha}$$

with

$$\phi(u) = \frac{\alpha\Gamma(\alpha u + 1)}{\Gamma(\alpha(u-1) + 1)} = \int_0^\infty (1 - e^{-ur})f(r) dr, \quad (3.2)$$

where

$$f(r) = \frac{e^{-r/\alpha}}{\Gamma(1-\alpha)(1 - e^{-r/\alpha})^{\alpha+1}}, \quad r > 0. \quad (3.3)$$

- (1) We start by applying the first part of Theorem 2.2. It is easy to check that the mapping $r \mapsto f(r)$ is decreasing on $(0, \infty)$ and thus, from Theorem 2.2 and using the recurrence relation $\Gamma(u+1) = u\Gamma(u)$, $u > 0$, we get that

$$\psi_1(u) = \frac{\Gamma(\alpha(u+1) + 1)}{\Gamma(\alpha u)},$$

which, after some easy calculations, yields

$$\psi_1(u) = \alpha\Gamma(\alpha+1)u + \int_{-\infty}^0 (e^{ur} - ur - 1) \frac{(\alpha+1)e^{(\alpha+1)r/\alpha}}{\alpha\Gamma(1-\alpha)(1 - e^{r/\alpha})^{\alpha+2}} dr. \quad (3.4)$$

Hence, from (3.1), we deduce that

$$I_{\psi_1} \stackrel{(d)}{=} e^{-\alpha}.$$

This result, up to a multiplicative constant, was actually obtained by Patie in [20], Theorem 4.1, where it is shown that the law of I_{ψ_1} is related to the distribution of the absorption time of the α -self-similar continuous-state branching process.

- (2) Moreover, let us define, as in Proposition 2.4,

$$\psi_2(u) = \frac{u}{u+1}\psi_1(u+1) = \alpha u \frac{\Gamma(\alpha(u+2) + 1)}{\Gamma(\alpha(u+1) + 1)}$$

and, after observing that

$$\frac{\partial}{\partial r} \left(\frac{e^{(\alpha+1)r/\alpha}}{(1 - e^{r/\alpha})^{\alpha+1}} \right) = \frac{(\alpha+1)e^{(\alpha+1)r/\alpha}}{\alpha(1 - e^{r/\alpha})^{\alpha+2}},$$

we obtain, from (2.3) and (3.4),

$$\psi_2(u) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha)}u + \int_{-\infty}^0 (e^{ur} - ur - 1) \frac{e^{(2\alpha+1)r/\alpha}}{\Gamma(1-\alpha)(1 - e^{r/\alpha})^{\alpha+2}} \left(\frac{2\alpha+1}{\alpha} - e^{r/\alpha} \right) dr.$$

Hence, from (1.5), we get, for any $n = 1, 2, \dots$,

$$\mathbb{E}[I_{\psi_2}^{-n}] = \alpha^{n-1} \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha)} \frac{\Gamma(\alpha n + \alpha + 1)}{\Gamma(2\alpha+1)} = \alpha^n \frac{\Gamma(\alpha n + \alpha + 1)}{\Gamma(\alpha+1)}$$

and, by moment identification, we have

$$\alpha I_{\psi_2} \stackrel{(d)}{=} G^{-\alpha}(\alpha+1),$$

where $G(a)$ stands for a gamma random variable of parameter $a > 0$. We deduce that, for any $\alpha \in (0, 1)$, the random variable $G^{-\alpha}(\alpha+1)$ is a positive self-decomposable random variable.

- (3) Next, we apply the converse part of Theorem 2.2 to ψ_2 . To this end, we introduce the subordinator with Laplace exponent ϕ_{-1} defined by

$$\phi_{-1}(u) = \frac{1}{u-1} \psi_2(u-1) = \frac{\psi_1(u)}{u} = \alpha \frac{\Gamma(\alpha(u+1)+1)}{\Gamma(\alpha u+1)}.$$

To get the Lévy–Khintchine representation of ϕ_{-1} , note, from (3.2), that

$$\phi_{-1}(u) = \phi(u+1) - \phi(1) + \phi(1).$$

That is, ϕ_{-1} is the Laplace exponent of the Esscher transform of ϕ killed at rate $\phi(1)$. Thus, using the expression (3.3),

$$\phi_{-1}(u) = \alpha \Gamma(\alpha+1) + \int_0^\infty (1 - e^{-ur}) e^{-r} f(r) dr.$$

We have, from (1.2),

$$\mathbb{E}[I_{\phi_{-1}}^n] = \alpha^{-n} \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(\alpha n + \alpha + 1)}, \quad n = 1, 2, \dots$$

In order to characterize the law of $I_{\phi_{-1}}$, let us denote by U an uniform random variable on $(0, 1)$ and by $S_1^{-\alpha}(\alpha)$ a random variable distributed according to the length-biased distribution of $S^{-\alpha}(\alpha)$, that is, for any measurable function $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, we have

$$\mathbb{E}[g(S_1^{-\alpha}(\alpha))] = \frac{\mathbb{E}[S^{-\alpha}(\alpha)g(S^{-\alpha}(\alpha))]}{\mathbb{E}[S^{-\alpha}(\alpha)]}.$$

Recalling that for any $n = 1, 2, \dots$,

$$\mathbb{E}[S^{-\alpha n}(\alpha)] = \frac{\Gamma(n+1)}{\Gamma(\alpha n + 1)},$$

we get, by taking the random variable U independent of $S_1^{-\alpha}(\alpha)$,

$$\begin{aligned}\mathbb{E}[U^n S_1^{-\alpha n}(\alpha)] &= \frac{\mathbb{E}[U^n] \mathbb{E}[S^{-\alpha(n+1)}(\alpha)]}{\mathbb{E}[S^{-\alpha}(\alpha)]} \\ &= \frac{\Gamma(\alpha+1) \Gamma(n+2)}{(n+1) \Gamma(\alpha n + \alpha + 1)} \\ &= \frac{\Gamma(\alpha+1) \Gamma(n+1)}{\Gamma(\alpha n + \alpha + 1)}.\end{aligned}$$

By moment identification, we deduce the following identity:

$$\alpha I_{\phi_{-1}} \stackrel{(d)}{=} U S_1^{-\alpha}(\alpha).$$

This yields the following factorization of the exponential law:

$$U S_1^{-\alpha}(\alpha) G^\alpha(\alpha+1) \stackrel{(d)}{=} \mathbf{e},$$

where the three random variables on the left-hand side are assumed to be independent.

On the other hand, Bertoin and Yor [6] also observed that

$$I_{\hat{\phi}} \stackrel{(d)}{=} \mathbf{e}^\alpha$$

with

$$\hat{\phi}(u) = u \frac{\Gamma(\alpha(u-1)+1)}{\Gamma(\alpha u+1)} = \int_0^\infty (1 - e^{-ur}) \frac{(1-\alpha)^2 e^{r/\alpha}}{\alpha \Gamma(\alpha+1) (e^{r/\alpha} - 1)^{2-\alpha}} dr.$$

Easily verifying that the density is decreasing, we obtain, appealing to obvious notation, that

$$\hat{\psi}_1(u) = u \frac{\Gamma(\alpha u+1)}{\alpha \Gamma(\alpha(u+1))},$$

which, after some easy manipulations, yields

$$\begin{aligned}\hat{\psi}_1(u) &= \Gamma^{-1}(\alpha+1)u \\ &\quad + \int_{-\infty}^0 (e^{ur} - ur - 1) \frac{(1-\alpha)^2 e^{-r/\alpha}}{\alpha^2 \Gamma(\alpha+1) (e^{-r/\alpha} - 1)^{2-\alpha}} \left(1 - \alpha + \frac{2-\alpha}{(1 - e^{r/\alpha})}\right) dr.\end{aligned}$$

Thus, from the identity (3.1) and Theorem 2.2, we deduce that

$$\mathcal{I}_{\hat{\psi}_1} \stackrel{(d)}{=} S(\alpha)^\alpha.$$

Hence, $S(\alpha)^\alpha$ is a positive self-decomposable random variable.

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